Solutions to Selected Problems from West, Installment One

1.2.17 Let G be the graph whose vertex set if the set of permutations of $\{1,\ldots,n\}$, with two permutations $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ adjacent if they differ by interchanging a pair of adjacent entries. Prove that G is connected.

Proof: Let $x = a_1,\ldots,a_n$ and $y = b_1,\ldots,b_n$ be vertices in G. Since x and y are permutations of $\{1,\ldots,n\}$, it follows that $y = x\phi$, where $\phi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a bijection. By a standard result from the lore of permutations, ϕ can be factored as a product of transpositions (pair interchanges); if k is the number of transpositions in the factorization of ϕ, then there is a path of length k connecting A to B. Thus G is connected. □

1.2.18 Let $G = (V,E)$, where $V = \{0,1\}^k$ and $xy \in E$ iff $H(x,y) = 2$, where $H(x,y)$ is the number of bits in which they disagree. How many components does G have?

Solution: Two. We discussed this one in class.

1.2.41 Let G be a connected graph on $n \geq 3$ vertices. Prove that G contains vertices x, y such that

(a) $G - \{x,y\}$ is connected, and

(b) $d(x,y) \leq 2$.

Proof: Let P be a path in G of longest length; let x be an endpoint of P, with predecessor (in a traversal of P ending at x) w. If $N(w) \subseteq V(P)$, let $y = w$. If not, let $y \in N(w) - V(P)$. Clearly, in the latter case, $N(y) \subseteq V(P)$, since otherwise we have a contradiction to our choice of P. The result follows. □

1.3.9 A baseball league has two divisions, each with 13 teams. Is it possible to schedule a season in which each team plays nine games within its division and four games against the other division?

Solution: Nope. Suppose, by way of contradiction, that we could arrange such a schedule. Construct a graph $G = (V,E)$, where V consists of the 26 teams and $xy \in E$ iff there is a game in which x plays y. Consider the subgraph H of G induced by either of the divisions. H is 9-regular on 13 vertices, but this is impossible. □

1.3.17 Let G be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot decrease the average degree.

Proof: (a) Let v be a vertex of degree $\Delta(G)$. The average degree in G is $\frac{2e}{n}$, while the average degree in $G-v$ is $\frac{2e-\Delta}{n-1}$. Since $n\Delta \geq 2e$, then $2ne - n\Delta \leq 2ne - 2e$, but then $\frac{2e-\Delta}{n-1} \leq \frac{2e}{n}$. A similar argument proves (b). □

1.3.23 Use induction to prove that the number of edges in Q_k is $k2^{k-1}$.

Proof: The result holds for $k=1$, i.e., $1(2^0)=1$. Assume that Q_{k-1} contains $(k-1)2^{k-2}$ edges. Consider Q_k. We know that Q_k comprises two copies of Q_{k-1} together with 2^{k-1} additional edges. It follows that the number of edges in Q_k is

$$\begin{aligned} 2e(Q_{k-1} + 2^{k-1} &= 2(k-1)2^{k-2} + 2^{k-1} \\ &= k2^{k-1} - 2^{k-1} + 2^{k-1} \\ &= k2^{k-1}, \end{aligned}$$

which is what we needed to show. □

1.3.47 Use induction on $n(G)$ to prove that every nontrivial loopless graph G has a bipartite subgraph H such that H has more than $e(G)/2$ edges.

Proof: K_2 is itself bipartite. Let G be a graph with k vertices, and assume that the result holds for graphs of smaller order. Let $v \in V(G)$. By the induction hypothesis, $G-v$ contains a bipartite subgraph $H' = (X, Y, F)$ containing more than $\frac{e(G-v)}{2}$ edges. If $N_G(v) \bigcap X > N_G(v) \bigcap Y$, add v to Y; otherwise add v to X to obtain the desired graph H. □

1.4.9 True or false: Every simple digraph either has two vertices with the same indegree or two vertices with the same outdegree.

Counterexample: Let $G = (V, E)$, where $V = \{a, b, c\}$ and $E = \{ab, ac, bc\}$. This is known as a transitive triple, for obvious reasons, and is an instance of a transitive tournament. Similar orientations of K_n are easily constructed for any choice of n. (Thanks to Pat Vincent for pointing out the folly of my previous “proof”, which has assumed its rightful place in the bit bucket.)

1.4.10 Prove that a digraph $G = (V, E)$ is strongly connected iff for every nontrivial partition of V as $V = S + T$ there is an edge from S to T.

Proof: First suppose that G is strongly connected. Let $V = S + T$ be any nontrivial partition of V. Let $x \in S$, $y \in T$. Since G is strongly connected, there is an x, y-path P in G. In a traversal of P, let z be the first vertex encountered in T, and let w be its predecessor on P. Clearly $w \in S$, and wz is the edge we want. Now suppose that G is not strongly connected. Choose x, y such that there is no x, y-path in G. Let $S = \{v | x \text{ reaches } v\}$, and let $T = V - S$. Note that $T \neq \emptyset$, since $y \in T$. Moreover, since x reaches every vertex in S but no vertex in T, there is no edge from S to T. $\square$

1.4.23 Prove that every graph G has an orientation D that is "balanced" at each vertex, i.e., that $|d_D^+(v) - d_D^-(v)| \leq 1$ for every $v \in V$.

Solution: There is an easy proof and a less-easy proof. I'll do them in that order.

Proof: (This is the easy one.) If G is Eulerian, then we're done: given $v \in V$, an Eulerian tour enters and leaves v an equal number of times, so if we orient the edges of G according to the directions taken on such a tour, we have $d_D^+(v) = d_D^-(v)$ for all $v \in V$. If not, then by the degree-sum theorem we know that there is an even number, say $2k$, of odd vertices in G. We can use k temporary edges to join pairs of odd vertices. The resulting graph, say G', is now Eulerian, and we proceed as before. Since $d_D^+(v) = d_D^-(v)$ for all v in G', it follows that when we erase the temporary edges we have $|d_D^+(v) - d_D^-(v)| \leq 1$ for all $v \in V$ in our orientation of G. $\square$

Proof: (The other one) Let G be a connected graph. If G is Eulerian, then we're done: given $v \in V$, an Eulerian tour enters and leaves v an equal number of times, so if we orient the edges of G according to the directions taken on such a tour, we have $d_D^+(v) = d_D^-(v)$ for all $v \in V$. Assume, then, that G is not Eulerian. We proceed by induction on E. Clearly K_1 and K_2 possess orientations D satisfying the desired inequality. Assume that G has $|E| > 1$ edges, and assume that all graphs with fewer edges possess orientations satisfying the inequality. There are two cases. First suppose that G has an edge $e = uv$ joining two odd vertices. By the induction hypothesis, we can let D be a balanced orientation of $G - e$. Note that, in $G - v$, both u and v are even vertices, so it must be that $d_D^+(u) = d_D^-(u)$ and $d_D^+(v) = d_D^-(v)$. We can then orient e in either direction to obtain an orientation of G, and the inequality still holds. Finally, suppose that no such edge exists. Then (since G is not Eulerian) every edge of G joins an even vertex to an odd vertex. Let $e = uv$ be such an edge; without loss of generality assume that u is odd. It follows that in $G - e$, u is even and v is odd. Once again, we apply the induction hypothesis to $G - e$ to obtain an orientation D. Since u is even, we know that $d_D^+(u) = d_D^-(u)$, so our eventual orientation of e won't cause trouble at u. But v is odd, so $|d_D^+(v) - d_D^-(v)| = 1$. If $d_D^+(v) > d_D^-(v)$, we orient e in the direction $u \to v$, while if $d_D^+(v) < d_D^-(v)$ we use the opposite orientation. In either case, joining u and v using the oriented e completes a balanced orientation of G. $\square$